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Chaos in two dimensions : an analytic study

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Abstract : A formulation is given to connect the spacing distribution with the distribution of the matrix elements of the Hamiltonian in two dimensions. The expression so obtained is used to see how the spacing distribution changes from Wigner's spacing distribution to that of Poisson We find that the key factor responsible for the deviation from Wigner's spacing distribution, is non-invariance of the form of the distribution of the Hamiltonian matrix elements under rotation.

Keywords : Wigner distribution, Poisson distribution, chaos

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Since the establishment of the connection between chaotic behaviour of a system and Wigner distribution for the spacing of nearest levels, there has been considerable interest in finding the key factor which is responsible for making a system either chaotic or integrable. In the past, this has been studied [1] by introducing Gaussian weight factors for the off-diagonal elements. From the generated spectrum of the Hamiltonian, moments of the spacing distribution of the nearest levels are obtained numerically and conclusions are drawn about level repulsion. In a different approach, small size band matrices are used [2,3] to see how Wigner repulsion changes when the off-diagonal elements of a random matrix are kept zero. The emphasis in both the studies has been the behaviour of the Wigner spacings distribution for small spacings with the change in the distribution of the off-diagonal elements.

In the present short note, we shall develop a formalism which connects the nearest neighbour level spacing distribution with those of the distribution of the Hamiltonian matrix elements. As shown later, this provides the key factor which determines the transition from Wigner distribution to Poisson distribution, the limit in which there is no level repulsion. To get analytic results, we study the problem in two dimensions.

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Let us consider a 2×2 real-symmetric Hamiltonian matrix H with diagonal elements H_{11}, H_{22} and off-diagonal elements H_{12}. Let ε_1, $\varepsilon_2 (\varepsilon_1 < \varepsilon_2)$ be its eigenvalues, then the distribution of the eigenvalues $P(E_1, E_2)$ is given by

$$P(E_1, E_2) = \delta(E_1 - \varepsilon_1)\delta(E_2 - \varepsilon_2). \tag{1}$$

Now if the elements H_{ij} have a given distribution given by $f(H_{11}, H_{22}, H_{12})$, then (1) has to be rewritten as

$$P(E_1, E_2) = \int \delta(E_1 - \varepsilon_1)\delta(E_2 - \varepsilon_2) f(H_{11}, H_{22}, H_{12}) \prod_{i \le j} dH_{ij}. \tag{2}$$

Realizing that $\varepsilon_1 + \varepsilon_2 = \mathrm{Tr}H$ and $\varepsilon_1^2 + \varepsilon_2^2 = \mathrm{Tr}H^2$, expression (2) can be rewritten as

$$P(E_1, E_2) = (E_2 - E_1)\int \delta(E_1 + E_2 - u)\,\delta\left(E_1^2 + E_2^2 - v\right) g(u, v)\,du\,dv, \tag{3}$$

where the function $g(u, v)$ is given by

$$g(u, v) = \int \delta(u - \mathrm{Tr}H)\,\delta(v - \mathrm{Tr}H^2) f(H_{11}, H_{22}, H_{12}) \prod_{i \le j} dH_{ij}. \tag{4}$$

By first using the transformation

$$h_{11} = \frac{1}{\sqrt{2}}(H_{11} + H_{22}), \qquad h_{22} = \frac{1}{\sqrt{2}}(H_{11} - H_{22})$$

and then $\quad h_{22} = \rho\cos\theta, \ \sqrt{2}H_{12} = \rho\sin\theta,$

in expression (4), we get

$$g(u, v) = \int_0^{2\pi} d\theta f\left(\frac{1}{2}\left[u + \sqrt{2v - u^2}\cos\theta\right], \ \frac{1}{2}\left[u - \sqrt{2v - u^2}\cos\theta\right], \right.$$
$$\left. \frac{1}{2}\sqrt{2v - u^2}\sin\theta\right). \tag{5}$$

We next obtain the spacing distribution $p(S)$ by taking the lower eigenvalue E_1 at $-\frac{S}{2}$ and the upper eigenvalue E_2 at $\frac{S}{2}$. Expressions (3) and (5) then give

$$p(S) = S\int_0^{2\pi} d\theta f\left(\frac{S}{2}\cos\theta, \ -S/2\cos\theta, \ \frac{S}{2}\sin\theta\right). \tag{6}$$

This expression connects the spacing distribution with the distribution of the matrix elements of the Hamiltonian H and can be used to study the repulsion of the two energy levels.

We first rederive Wigner's spacing distribution by taking the function f to be

$$f = \exp(-\mathrm{Tr}H^2). \tag{7}$$

Expressions (6) and (7) immediately give

$$p(S) = S \exp\left(-\frac{S^2}{2}\right), \tag{8}$$

which is the well-known Wigner's spacing distribution having the repulsion S when S is small. In writing expressions (7) and (8) we have not normalized them to unity. This can easily be done, but since we are interested in studying the repulsion phenomenon we shall leave these functions as unnormalized.

We would next like to see what form of the function f gives Poisson distribution, the distribution in which repulsion is absent. Let us take f to be given by

$$f = \left[\exp - \left(|H_{11}| + |H_{22}|\right)\right] \delta(H_{12}), \tag{9}$$

then expressions (6) and (9) give

$$p(S) = \exp(-S), \tag{10}$$

which is Poisson distribution. For this distribution if $S \to 0$, $p(S) \to$ constant and so the distribution is peaked at $S = 0$. This behaviour for integrable systems has also been given in the studies carried out by McDonald and Kaufman [4].

The absence of level repulsion also occurs if we take f to be given by

$$f = \left[\exp - \left(H_{11}^2 + H_{22}^2\right)\right] \delta(H_{12}), \tag{11}$$

then (6) and (11) give

$$p(S) = \exp\left(-S^2/2\right). \tag{12}$$

This is not strictly Poisson distribution, but a distribution which has the characteristic of Poisson distribution for small S, namely the peaking of the distribution at $S = 0$. Thus as earlier [1], if the off-diagonal element is taken to be zero, Wigner's level repulsion vanishes.

The interesting result now to note is that the level repulsion also vanishes if one of the diagonal elements, say H_{11} is taken as $\delta(H_{11})$. Thus if the function f is taken to be

$$f = \left[\exp - \left(H_{22}^2 + H_{12}^2\right)\right] \delta(H_{11}),$$

then $p(S)$ is again given by

$$p(S) = \exp\left(-\frac{S^2}{4}\right).$$

Before we give the key factor, let us consider one more distribution which gives $P(S)$ between Wigner's and Poisson distribution in the limit of $S \to 0$. Let us assume that f is given by

$$f = \exp - \left(H_{11}^2 + H_{22}^2 + \frac{|H_{12}|}{\sigma} \right), \tag{13}$$

where σ is a parameter.

Putting this in expression (6) we get

$$p(S) = 2S \left[\exp\left(-\frac{S^2}{2} \right) \right] \int_0^{\pi} d\theta \exp\left[\frac{S^2}{2} \sin^2\theta - \frac{S}{2\sigma} \sin\theta \right], \tag{14}$$

if σ is small, this gives [5]

$$p(S) = 2\pi S \exp\left(-\frac{S^2}{2} \right) I_0\left(\frac{S}{2\sigma} \right), \tag{15}$$

where I_0 is the modified Bessel function. Using the asymptotic form [5] of I_0 when $\sigma \to 0$, we get

$$p(S) = 2\sqrt{\sigma\pi S} \exp\left(-\frac{S^2}{2} + \frac{S}{2\sigma} \right), \tag{16}$$

which has level repulsion in between Wigner's and Poisson distribution as $S \to 0$.

We can also show that the following form of the matrix elements distribution leads to spacing distribution which interpolate between the Poisson and Wigner distribution. This form is given by

$$f = \exp - \left(|H_{11}|^{\lambda} + |H_{22}|^{\lambda} + \frac{|H_{12}|^{r}}{\sigma} \right),$$

where λ, r and σ are parameters.

This, *e.g.* leads to a distribution of spacing when $\sigma \to 0$ of the form

$$p(s) = s^r \exp\left[-2\left(\frac{s}{2} \right)^{\lambda} \right].$$

By appropriately choosing r, λ, one can get, *e.g.*, Brody distribution [6].

From the above examples of various forms of f, we find that the key factor which determines the transition from Wigner's to Poisson distribution is the invariance of the form of f under rotation. Any function f which is not invariant under rotation will give repulsion different than Wigner's distribution *e.g.* not only $\delta(H_{12})$ but also $\delta(H_{11})$. The behaviour for small S is such that the spacing is peaked at $S = 0$.

We have shown how to connect the spacing distribution to the distribution of the matrix elements of the Hamiltonian in two dimensions. From this relation, we find that non-invariance of the distribution of the Hamiltonian matrix elements gives deviations from Wigner's spacing distribution.

The present study can further be extended to three dimensions, where one has to introduce the distribution $g(u, v, w)$ defined by

$$g(u, v, w) = \int \delta(u - \mathrm{Tr}H)\, \delta(v - \mathrm{Tr}H^2)\, \delta(w - \mathrm{Tr}H^3) \times P(\{H_{ij}\}) \prod_{i \le j} dH_{ij} \tag{17}$$

which can then be used to connect the spacing distribution to the distribution of the matrix elements of the Hamiltonian.

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